NOAA Technical Report ERL 451–ETL 66

The Assumption of Joint Gaussian Velocities as Applied to the Pressure Structure Function

Reginald J. Hill

December 1994

U.S. DEPARTMENT OF COMMERCE
National Oceanic and Atmospheric Administration
Environmental Research Laboratories

The Assumption of Joint Gaussian Velocities as Applied to the Pressure Structure Function

Reginald J. Hill

Environmental Technology Laboratories
Boulder, Colorado

December 1994

NOTICE

Mention of a commercial company or product does not constitute an endorsement by NOAA/ERL. Use for publicity or advertising purposes, of information from this publication or concerning proprietary products or the tests of such products, is not authorized.

CONTENTS

The Assumption of Joint Gaussian Velocities as Applied to the Pressure Structure Function

Reginald J. Hill

ABSTRACT. The 1951 theory of the pressure structure function used the assumption of joint Gaussian velocities to obtain tractable results. That theory has recently been replaced by a new theory that does not use this assumption. The relationships between formulas from the two theories are given. It is shown that combining the new theory with the joint Gaussian assumption (JGA) produces the 1951 theory. Formulas for the pressure structure function and the pressure-gradient correlation are given, as are asymptotic formulas for the inertial and dissipation ranges. The values of constants and parameters in the new theory are evaluated using the JGA. The incompatibility of intermittency with the JGA causes the 1951 theory to have an opposite intermittency correction to inertial-range exponents as compared with the new theory.

1. INTRODUCTION

A new relationship between the pressure structure function $D_p(r)$ and the fourth-order velocity structure function $D_{ijkl}(\vec{r})$ was derived by Hill (1993) and Hill and Wilczak (1995), where the separation vector between two points is denoted by $\vec{r}$, and $r = |\vec{r}|$. The only assumptions they used are incompressibility, local isotropy, and the Navier-Stokes equation. Previously, Obukhov (1949) and Obukhov and Yaglom (1951) obtained a relationship between $D_p(r)$ and the second-order velocity structure function $D_{ij}(\vec{r})$ by use of the additional assumption that velocity derivatives at different positions have a joint Gaussian probability density function (PDF). Yaglom (1949) used Obukhov's (1949) theory to obtain the mean-squared pressure gradient. Obukhov and Yaglom (1951) derived the pressure gradient correlation. The assumption that velocities at different positions have a joint Gaussian PDF was used by Batchelor (1951) to relate the fourth-order velocity correlation $R_{ijkl}(\vec{r})$ to $D_{ij}(\vec{r})$ and thereby relate $D_p(r)$ to $D_{ij}(\vec{r})$. We clarify here the relationship between the 1951 theory and the theories of Hill (1993) and Hill and Wilczak (1995).

The limits of validity of the joint Gaussian assumption (JGA) as applied to $D_p(r)$ need to be established, as well as, on the basis of the JGA, the numerical values of the constants introduced by Hill (1993). The data by George et al. (1984) suggest that the JGA gives a good estimate (within a factor of 2) of $D_p(r)$ in the inertial range; the difficulty of making inertial-range pressure measurements continues to prevent a detailed investigation. However, the new theory allows one to compare the predictions of the JGA with those of the new theory from measurements of the velocity vector; pressure measurements are not needed for this purpose. Specifically, $D_{ij}(\vec{r})$ and $D_{ijkl}(\vec{r})$ must be measured simultaneously, preferably for all relevant spacings r. We present results from the JGA that are comparable to results of the new theory. These results can be the basis of an experimental investigation. We expect the JGA-based theory by Batchelor (1951) and Obukhov and Yaglom (1951) to be less accurate for decreasing r/L, where L is a scale of the energy-containing range. Also, for r within the inertial range, smaller values of r/L are attainable for greater Reynolds numbers.

That velocity differences are non-Gaussian is characteristic of intermittency. However, the JGA formulas can include an intermittency effect simply by substituting the form of $D_{ij}(\vec{r})$ that includes the intermittency correction, and hence the JGA formulas can contain the intermittency exponent μ. In a sense, the best available formulas for $D_{ij}(\vec{r})$ should be used in the JGA formulas. On the other hand, setting $\mu = 0$ is more consistent with the JGA. We use equality (=) to indicate that a JGA formula includes the possibility that $\mu \neq 0$, and we use $\simeq$ when a formula is simplified to the case $\mu = 0$.

Assumptions needed for our results are obvious from our notation. Definitions need no assumptions and are distinguished by use of the identity symbol ($\equiv$) rather than equality (=). If the vector separation $\vec{r}$ appears on the right side of an equation, then local homogeneity is assumed. If the spacing r, or wave number k, or wave vector component k_1 appears on the right side of an equation, then local isotropy is assumed. If these rules are violated, then we specifically state the assumption, e.g., use of homogeneity as distinguished from local homogeneity or use of isotropy as distinguished from local isotropy.

2. GENERAL FORMULAS FROM THE JOINT GAUSSIAN ASSUMPTION

The symbols u_i, P, and ρ denote velocity component, pressure, and density, respectively. Primed and unprimed quantities are taken at the distinct spatial points $\vec{x}'$ and $\vec{x}$, and the separation vector between these points is denoted by $\vec{r}$, which has magnitude r. We use the convention that summation is implied by repeated Roman indices but not by repeated Greek indices; angle brackets denote averaging; subscripts following a vertical bar denote differentiation. Following the notation by Hill (1993), a vertical bar on the outside of an average denotes differentiation with respect to the components of $\vec{r}$, whereas a vertical bar with subscripts within an average denotes differentiation with respect to components of $\vec{x}$ if the quantity being differentiated is unprimed and with respect to components of $\vec{x}'$ if the quantity is primed.

We define the quantities

$$D_{ij}(\vec{r}) \equiv \left\langle \left(u_i - u_i'\right)\left(u_j - u_j'\right) \right\rangle = 2\left[\sigma_{ij} - R_{ij}(\vec{r})\right],$$

where we use isotropy to simplify the relationship of $D_{ij}(\vec{r})$ to the velocity correlation $R_{ij}(\vec{r})$, which is, in turn, defined by

$$R_{ij}(\vec{r}) \equiv \left\langle u_i u_j' \right\rangle,$$

and the velocity covariance is

$$\sigma_{ij} \equiv \langle u_i u_j \rangle = R_{ij}(0).$$

Further definitions are

$$D_{ijkl}(\vec{r}) \equiv \left\langle \left(u_i - u_i'\right)\left(u_j - u_j'\right)\left(u_k - u_k'\right)\left(u_l - u_l'\right)\right\rangle$$

$$R_{ijkl}(\vec{r}) \equiv \left\langle u_i u_j u_k' u_l' \right\rangle \qquad (1)$$

$$Q(r) \equiv \left\langle (u_i u_j)_{|ij} (u_k' u_l')_{|kl} \right\rangle = \left\langle u_{i|j} u_{j|i} u_{k|l}' u_{l|k}' \right\rangle \qquad (2a)$$

$$= \frac{1}{6} D_{ijkl}(\vec{r})_{|ijkl} \qquad (2b)$$

$$= R_{ijkl}(\vec{r})_{|ijkl}, \qquad (2c)$$

where (2b) is derived by Hill and Wilczak (1995) on the basis of local homogeneity and (2c) requires homogeneity, and

$$D_P(r) \equiv \frac{1}{\rho^2} \left\langle \left(P - P'\right)^2 \right\rangle$$

$$= \frac{1}{3r} \int_0^r \left(y^4 - 3ry^3 + 3r^2y^2\right) Q(y)\, dy + \frac{r^2}{3} \int_r^\infty y\, Q(y)\, dy. \qquad (3)$$

The pressure-gradient correlation is

$$A_{ij}(\vec{r}) \equiv \frac{1}{\rho^2} \left\langle P_{|i} P_{|j}' \right\rangle = \frac{1}{2} D_P(r)_{|ij}.$$

The mean-squared pressure gradient is

$$\chi \equiv \frac{1}{\rho^2} \left\langle P_{|i} P_{|i} \right\rangle = A_{ii}(0).$$

3

The incompressibility condition gives

$$D_{ij}(\vec{r})_{|i} = D_{ij}(\vec{r})_{|j} = 0 \tag{4a}$$

$$R_{ij}(\vec{r})_{|i} = R_{ij}(\vec{r})_{|j} = 0 , \tag{4b}$$

where (4a) requires local homogeneity and (4b) requires homogeneity.

Applying the JGA to (1) and simplifying using isotropy gives (Hill, 1993)

$$R_{ijkl}^{JG}(\vec{r}) = \sigma_{ij}\sigma_{kl} + R_{ik}(\vec{r})R_{jl}(\vec{r}) + R_{il}(\vec{r})R_{jk}(\vec{r}) \tag{5a}$$

$$R_{ijkl}^{JG}(\vec{r}) = \left[\sigma_{ij}\sigma_{kl} + \sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk}\right]$$

$$- \frac{1}{2}\left[\sigma_{ik}D_{jl}(\vec{r}) + \sigma_{jl}D_{ik}(\vec{r}) + \sigma_{il}D_{jk}(\vec{r}) + \sigma_{jk}D_{il}(\vec{r})\right]$$

$$+ \frac{1}{4}\left[D_{ik}(\vec{r})D_{jl}(\vec{r}) + D_{il}(\vec{r})D_{jk}(\vec{r})\right]. \tag{5b}$$

We use superscript *JG* to denote that the formula for a quantity derives from the JGA. The formula (5a) was used by Batchelor (1951), but it obscures the nature of the approximation, whereas (5b) is revealing. Taking the fourth-order divergence of (5b), we see that the first term in (5b) vanishes because it is constant and the second term vanishes because of incompressibility (4b), and therefore only the last term in (5b) contributes to $D_P^{JG}(r)$ as given in (3). As r/L decreases in the inertial range, the last term in (5b) is smaller than the second term by a factor of the order $(r/L)^{2/3}$ and smaller than the first term by a factor of the order $(r/L)^{4/3}$. Thus, the last term in (5b), which is the only term contributing to $D_p(r)$, is asymptotically much smaller than the other terms in (5b). By asymptotic, we mean that r/L approaches zero for some r within the inertial range; this requires asymptotically large Reynolds numbers. We see that it is really the combination of the JGA and incompressibility that reduces the large and extraneous values of $R_{ijkl}(\vec{r})$ to a tractable result.

It seems that the JGA is inaccurate for the purpose of obtaining $D_p(r)$ from $R_{ijkl}(\vec{r})$ because we require the JGA to be so accurate that the last and asymptotically smallest term in (5b) is accurate. We now show that this is not a valid criticism of the JGA. As shown by Hill (1993) and Hill and Wilczak (1995), $D_p(r)$ can be obtained using (2b) and (3); that is, $D_{ijkl}(\vec{r})$ is the appropriate statistic, and

$$D_{ijkl}^{JG}(\vec{r}) = D_{ik}(\vec{r})D_{jl}(\vec{r}) + D_{il}(\vec{r})D_{jk}(\vec{r}) + D_{ij}(\vec{r})D_{kl}(\vec{r}). \tag{6}$$

4

There are no asymptotically very large terms in (6) that cancel when the fourth-order divergence is performed. Taking the fourth-order divergence of either (5b) or (6) and using incompressibility (4a,b), and then substituting the result in (2b) and (2c) gives

$$Q^{JG}(r) = R^{JG}_{ijkl}(\vec{r})_{|ijkl} = \frac{1}{6} D^{JG}_{ijkl}(\vec{r})_{|ijkl} = \frac{1}{2} D_{ij}(\vec{r})_{|kl} D_{kl}(\vec{r})_{|ij}.$$

Thus, (5b) gives the same $Q^{JG}(r)$ as does (6), and (5b) therefore produces the same $D^{JG}_P(r)$ from (3) as does (6). The plausibility of applying the JGA to $D_P(r)$ requires that (6) be accurate; (5b) need not be accurate on the order of its asymptotically smallest term. Under the JGA, the pressure structure function and all related quantities (i.e., mean-squared pressure gradient, pressure-gradient correlation, asymptotic formulas) are obtained by substituting (6) into the formulas by Hill (1993) and Hill and Wilczak (1995).

We now use the preferred Cartesian coordinate system having its 1-axis along the separation vector $\vec{r}$. Subscripts α and β can take on values 1, 2, or 3, but the subscripts λ and γ are limited to 2 and 3. In the preferred coordinate system, the locally isotropic tensors $D_{ij}(\vec{r})$ have the nonzero components $D_{11}(r)$ and $D_{\lambda\lambda}(r)$. Likewise, $A_{ij}(\vec{r})$ has nonzero components $A_{11}(r)$ and $A_{\lambda\lambda}(r)$, and $D_{ijkl}(\vec{r})$ has nonzero components of the form $D_{\alpha\alpha\beta\beta}(r)$; the special case $\alpha = \beta$ gives the components $D_{\alpha\alpha\alpha\alpha}(r)$. From (6), the components of $D^{JG}_{ijkl}(\vec{r})$ that are nonzero under local isotropy are related to second-order structure-function components by

$$D^{JG}_{\alpha\alpha\beta\beta}(r) = \Delta_{\alpha\beta} D_{\alpha\alpha}(r) D_{\beta\beta}(r), \tag{7}$$

where

$$\Delta_{\alpha\beta} = \begin{cases} 3, & \text{for } \alpha = \beta, \\ 1, & \text{for } \alpha \neq \beta. \end{cases}$$

Under local isotropy and with use of the preferred coordinate system, the incompressibility condition (4a) simplifies to

$$D^{(1)}_{11}(r) - \frac{2}{r}\left[D_{\lambda\lambda}(r) - D_{11}(r)\right] = 0. \tag{8}$$

The superscript in parentheses indicates the order of differentiation with respect to r. A combination of components of $D_{ijkl}(\vec{r})$ that occurs frequently in formulas by Hill and Wilczak (1995) is

5

$$A_D(r) \equiv D_{1111}(r) + D_{\lambda\lambda\lambda\lambda}(r) - 6 D_{11\gamma\gamma}(r). \tag{9}$$

Substituting (7) into (9) gives

$$A_D^{JG}(r) = 3 \left\{ [D_{11}(r)]^2 + [D_{\lambda\lambda}(r)]^2 - 2 D_{11}(r) D_{\gamma\gamma}(r) \right\}$$

$$= 3 [D_{\lambda\lambda}(r) - D_{11}(r)]^2 = \frac{3}{4} r^2 \left[D_{11}^{(1)}(r) \right]^2, \tag{10}$$

where the last formula follows from incompressibility (8).

Substituting (7) and (8) in the formula for $Q(r)$ given by Hill (1993) and Hill and Wilczak (1995), we obtain

$$Q^{JG}(r) = 2 \left[D_{11}^{(2)}(r) \right]^2 + 2 D_{11}^{(1)}(r) D_{11}^{(3)}(r)$$

$$+ \frac{10}{r} D_{11}^{(1)}(r) D_{11}^{(2)}(r) + \frac{3}{r^2} \left[D_{11}^{(1)}(r) \right]^2. \tag{11}$$

This is the same as Batchelor's (1951) equation (5.3), which serves as a check on the derivation of $Q(r)$ by Hill (1993). We can obtain $D_P^{JG}(r)$ by substituting (11) in (3) and integrating by parts. However, it is easier to obtain the formula for $D_P^{JG}(r)$ by substituting (7) in the formula for $D_P(r)$ in Hill (1993) and Hill and Wilczak (1995), yielding

$$D_P^{JG}(r) = - [D_{11}(r)]^2 + \frac{4}{3} r^2 \int_r^\infty y^{-3} A_D^{JG}(y) \, dy$$

$$+ 4 \int_0^r y^{-1} \left\{ [D_{\lambda\lambda}(y)]^2 - D_{11}(y) D_{\gamma\gamma}(y) \right\} dy. \tag{12}$$

Using (8) in (12) gives

$$D_P^{JG}(r) = \int_0^r y \left[D_{11}^{(1)}(y) \right]^2 dy + r^2 \int_r^\infty y^{-1} \left[D_{11}^{(1)}(y) \right]^2 dy. \tag{13}$$

It is easy to show that (13) is the same as Batchelor's (1951) equation (6.4) with a correction of the sign in (6.4). This serves as a check on the integration by parts that was performed by Hill (1993) to derive $D_p(r)$. Obukhov and Yaglom (1951) gave a more complex equation than (13) because they did not simplify their result using integration by parts. Similarly,

6

substituting (7) in the formula by Hill (1993) and Hill and Wilczak (1995) for the spectrum from data along a line, we obtain

$$\Psi_P^{JG}(k_1) = \frac{8}{3\pi} \int_0^\infty \left[\frac{\sin(k_1 r)}{(k_1 r)^3} - \frac{\cos(k_1 r)}{(k_1 r)^2} \right] A_D^{JG}(r)\, dr, \tag{14}$$

where k_1 is the component of the wave vector along the 1-axis. Under the JGA, the formula for the mean-squared pressure gradient by Hill (1993) and Hill and Wilczak (1995) becomes

$$\chi^{JG} = 4 \int_0^\infty r^{-3}\, A_D^{JG}(r)\, dr = 3 \int_0^\infty r^{-1} \left[D_{11}^{(1)}(r) \right]^2 dr, \tag{15}$$

which is the same as Batchelor's (1951) equation (5.7). In the JGA, the pressure-gradient correlation is

$$A_{\lambda\lambda}^{JG}(r) = \int_r^\infty y^{-1} \left[D_{11}^{(1)}(y) \right]^2 dy \tag{16a}$$

$$A_{11}^{JG}(r) = A_{\lambda\lambda}^{JG}(r) - \left[D_{11}^{(1)}(r) \right]^2. \tag{16b}$$

The result (16a,b) follows from use of (7) and (8) in the formulas given by Hill (1993) or Hill and Wilczak (1995) for $A_{\lambda\lambda}(r)$ and $A_{11}(r)$. Formulas (16a,b) are much simpler than the corresponding formulas by Obukhov and Yaglom (1951) because Obukhov and Yaglom did not simplify their results using integration by parts.

3.   ASYMPTOTIC FORMULAS FOR $D_{ij}(\vec{r})$ AND $D_{ijkl}^{JG}(\vec{r})$

In this section, we state the asymptotic formulas for the inertial and viscous ranges of $D_{\alpha\alpha}(r)$ and use these formulas to determine the corresponding asymptotic formulas for $D_{\alpha\alpha\beta\beta}^{JG}(r)$. This allows us to give the JGA predictions for numerical values of the universal constants defined by Hill (1993). These results are used in Sec. 4 to determine asymptotic formulas for $D_P^{JG}(r)$ and related quantities.

For the inertial range, we have

$$D_{\alpha\alpha}(r) = C_\alpha \varepsilon^{2/3} r^g, \tag{17}$$

where

$$g = \frac{2}{3} + \frac{\mu}{9}$$
(18)

and ε is the energy dissipation rate. Of course, Kolmogorov constants C_α have units of length to the $-\mu/9$ power (if $\mu \neq 0$) and a dependence on turbulence macrostructure; they are related by substitution of (17) in (8) to obtain

$$C_\lambda = \left(1 + \frac{g}{2}\right) C_1 \approx \frac{4}{3} C_1 .$$
(19)

Recall that we use $\approx$ to mean that the formula is simplified to the case $\mu = 0$. Yaglom (1981) reviewed values of C_1 and recommended $C_1 = 2$. Values between 2.1 and 2.5 were recently obtained by Praskovsky and Oncley (1994).

The viscous-range formula is

(20)
$$D_{\alpha\alpha}(r) = d_\alpha r^2 ,$$

where

$$d_\alpha \equiv \left\langle \left(u_{\alpha|1}\right)^2 \right\rangle .$$
(21)

Substituting (20) in (8) gives

$$d_\lambda = 2 d_1 .$$
(22)

The inner scales ℓ_α are defined by equating the inertial- and viscous-range formulas for $D_{\alpha\alpha}(r)$, namely, (17) and (20), at $r = \ell_\alpha$, which gives

$$\ell_\alpha \equiv \left(C_\alpha \varepsilon^{2/3} / d_\alpha\right)^{1/(2-g)} \approx \left(C_\alpha \varepsilon^{2/3} / d_\alpha\right)^{3/4}$$
(23)

8

Thus, using (19), (22), and (23), we have

$$\ell_\lambda / \ell_1 = \left(\frac{C_\lambda}{C_1} \frac{d_1}{d_\lambda} \right)^{1/(2-g)} = \left[\frac{1}{2} \left(1 + \frac{g}{2} \right) \right]^{1/(2-g)} = 0.74 \qquad (24a)$$

$$\approx \left(\frac{2}{3} \right)^{3/4} \approx 0.74 . \qquad (24b)$$

Numerical values such as (24a) use $\mu = 0.25$, as recommended by Sreenivasan and Kailasnath (1993); of course, (24b) uses $\mu = 0$. Also, from (23),

$$\ell_1 \approx \left(15 C_1 \right)^{3/4} (\nu^3/\varepsilon)^{1/4} \approx 13\eta , \qquad (25)$$

where ν is kinematic viscosity, η is the Kolmogorov microscale, and we used $\varepsilon = 15\nu d_1$.

We need a specific definition of the scale L of the energy-containing range. Therefore, we define the scale L to be the spacing at which the inertial-range formula for $D_{11}(r)$ equals $D_{11}(\infty)$; that is,

$$C_1 \varepsilon^{2/3} L^g = 2\sigma_{11} .$$

Then, for r in the inertial range,

$$D_{11}(r)/\sigma_{11} = 2 (r/L)^g . \qquad (26)$$

For the inertial range of $D_{\alpha\alpha\beta\beta}(r)$, Hill (1993) gave

$$D_{\alpha\alpha\beta\beta}(r) = C_{\alpha\beta} \varepsilon^{4/3} r^q , \qquad (27)$$

where

$$q = \frac{4}{3} - \frac{2\mu}{9} . \qquad (28)$$

9

Substituting (17) in (7) gives

$$D_{\alpha\alpha\beta\beta}^{JG}(r) = \Delta_{\alpha\beta} C_\alpha C_\beta \, \varepsilon^{4/3} \, r^{2g} . \tag{29}$$

Comparing (29) with (18) and (27) gives the JGA correspondence

$$C_{\alpha\beta}^{JG} = \Delta_{\alpha\beta} C_\alpha C_\beta \tag{30}$$

and

$$q^{JG} = 2g = \frac{4}{3} + \frac{2\mu}{9} . \tag{31}$$

The last terms in (28) and (31) have opposite signs. Therefore, the dependence on μ of $D_{\alpha\alpha\beta\beta}^{JG}(r)$ and $D_P^{JG}(r)$ is opposite to the dependence on μ of $D_{\alpha\alpha\beta\beta}(r)$ and $D_P(r)$. Use of the JGA in combination with the intermittency-theory (and, presumably, most accurate) formula for $D_{ij}(\vec{r})$ causes less accurate pressure statistics than does the combination of the JGA and the classic theory of $D_{ij}(\vec{r})$ given by Kolmogorov (1941).

We define the flatness factor as

$$F \equiv D_{1111}(r) / \left[D_{11}(r) \right]^2 \propto r^{-4\mu/9} \tag{32a}$$

$$\propto C_{11} / C_1^2 . \tag{32b}$$

Following the definition $H_{\alpha\beta} \equiv C_{\alpha\beta} / C_{11}$ by Hill (1993), we have from (30),

$$H_{\alpha\beta}^{JG} = C_{\alpha\beta}^{JG} / C_{11}^{JG} = \Delta_{\alpha\beta} C_\alpha C_\beta / 3 \, C_1^2 . \tag{33}$$

Using (19), we have

$$H_{\lambda\lambda}^{JG} = \left(1 + \frac{g}{2} \right)^2 \approx \frac{16}{9} \tag{34a}$$

and

$$H_{1\lambda}^{JG} = \frac{1}{3} \left(1 + \frac{g}{2} \right) \approx \frac{4}{9} . \tag{34b}$$

10

For the viscous range of $D_{\alpha\alpha\beta\beta}(r)$, the formula is

$$D_{\alpha\alpha\beta\beta}(r) = d_{\alpha\beta} \, r^4, \tag{35}$$

where

$$d_{\alpha\beta} \equiv \left\langle \left(u_{\alpha|1}\right)^2 \left(u_{\beta|1}\right)^2 \right\rangle. \tag{36}$$

Substituting (20) in (7) gives

$$D_{\alpha\alpha\beta\beta}^{JG}(r) = \Delta_{\alpha\beta} \, d_\alpha \, d_\beta \, r^4, \tag{37}$$

so

$$d_{\alpha\beta}^{JG} = \Delta_{\alpha\beta} \, d_\alpha \, d_\beta. \tag{38}$$

The ratio of derivatives in (36) is defined by Hill (1993) as $\Lambda_{\alpha\beta} \equiv d_{\alpha\beta}/d_{11}$; in the JGA, we have

$$\Lambda_{\alpha\beta}^{JG} = \frac{d_{\alpha\beta}^{JG}}{d_{11}^{JG}} = \Delta_{\alpha\beta} \, d_\alpha d_\beta / 3 d_1^2. \tag{39}$$

Thus, from (22),

$$\Lambda_{\lambda\lambda}^{JG} = \left(d_\lambda/d_1\right)^2 = 4 \tag{40a}$$

and

$$\Lambda_{1\lambda}^{JG} = d_\lambda/3d_1 = \frac{2}{3}. \tag{40b}$$

The derivative kurtosis is defined by

$$K_{11} \equiv d_{11}/d_1^2. \tag{41}$$

11

Hill (1993) defined the universal constant $h_{\alpha\beta}$ by

$$h_{\alpha\beta} \equiv \Lambda_{\alpha\beta}^{(2-q)/(4-q)} H_{\alpha\beta}^{2/(4-q)} .$$

This quantity $h_{\alpha\beta}$ appears in simplified formulas for the mean-squared pressure gradient χ. Using (31), we have the JGA result,

$$h_{\alpha\beta}^{JG} = \Lambda_{\alpha\beta}^{JG(1-g)/(2-g)} H_{\alpha\beta}^{JG1/(2-g)} \tag{42}$$

From (42), we obtain

$$h_{\lambda\lambda}^{JG} \propto 4^{1/4}\left(\frac{16}{9}\right)^{3/4} = 2.177 \tag{43a}$$

and

$$h_{1\lambda}^{JG} \propto \left(\frac{2}{3}\right)^{1/4}\left(\frac{4}{9}\right)^{3/4} = 0.492 . \tag{43b}$$

The definition of the inner scale $\ell_{\alpha\beta}$ of $D_{\alpha\alpha\beta\beta}(r)$ is obtained by equating (27) and (35) at $r = \ell_{\alpha\beta}$. Either from this definition of $\ell_{\alpha\beta}$ or by equating (29) and (37) at $r = \ell_{\alpha\beta}^{JG}$, we have

$$\ell_{\alpha\beta}^{JG} = \left(C_\alpha C_\beta \varepsilon^{4/3}/d_\alpha d_\beta\right)^{1/(4-2g)}$$

$$= (\ell_\alpha \ell_\beta)^{1/2} ; \tag{44}$$

for instance, $\ell_{11}^{JG} = \ell_1$. From (24a,b) and (44), we have

$$\frac{\ell_{\lambda\lambda}^{JG}}{\ell_{11}^{JG}} = \frac{\ell_\lambda}{\ell_1} \propto 0.74 \tag{45a}$$

and

$$\frac{\ell_{1\lambda}^{JG}}{\ell_{11}^{JG}} = \left(\frac{\ell_\lambda}{\ell_1}\right)^{1/2} \propto 0.86 . \tag{45b}$$

12

The quantity $Q(0)$ is important for consideration of the pressure-gradient correlation and the transition between inertial and viscous ranges of $D_p(r)$. Hill (1993) defined

$$h_Q \equiv 1 + \frac{1}{3}\Lambda_{\lambda\lambda} - 3\Lambda_{1\gamma} \tag{46a}$$

such that $Q(0) = 60\, h_Q\, d_{11}$. In the JGA, we have

$$h_Q^{JG} = 1 + \frac{1}{3}\Lambda_{\lambda\lambda}^{JG} - 3\Lambda_{1\gamma}^{JG} \tag{46b}$$

$$= 1 + \frac{4}{3} - 2 = \frac{1}{3}, \tag{46c}$$

where (46c) follows from (46b) by substituting (40a,b). That is, in the JGA we have

$$Q^{JG}(0) = 60\, d_1^{\,2}. \tag{47}$$

4.   ASYMPTOTIC FORMULAS FOR $D_P^{JG}(r)$ AND RELATED QUANTITIES

Next, we use the definitions and numerical constants in Sec. 3 to investigate asymptotic formulas for $D_p(r)$, χ, $A_{ij}(\vec{r})$, and the relevant length scales of transition between asymptotic ranges. The dissipation-range formula for $D_P^{JG}(r)$ is obtained by power series expansion of (13), yielding

$$D_P^{JG}(r) = \frac{1}{3}\chi^{JG} r^2 - d_1^2\, r^4 + \ldots. \tag{48a}$$

$$= \frac{1}{3}\chi^{JG} r^2 - \frac{Q^{JG}(0)}{60} r^4 + \ldots , \tag{48b}$$

where we used (21) and (47). The form (48b) shows that the dissipation-range formula for $D_P^{JG}(r)$ also follows from the analogous formula for $D_p(r)$ given by Hill (1993) and Hill and Wilczak (1995) with χ and $Q(0)$ replaced by χ^{JG} and $Q^{JG}(0)$.

13

For the inertial range of $D_P^{JG}(r)$, we use (17) in (12), yielding

$$D_P^{JG}(r) = \left\{ g(1+g) \left[D_{11}(r) \right]^2 + 2 \left[D_{\lambda\lambda}(r) \right]^2 - 2(1+g) D_{11}(r) D_{\gamma\gamma}(r) \right\} \Big/ g(1-g) \qquad (49a)$$

$$\approx 5 \left[D_{11}(r) \right]^2 + 9 \left[D_{\lambda\lambda}(r) \right]^2 - 15 D_{11}(r) D_{\gamma\gamma}(r) \qquad (49b)$$

$$\approx \left[D_{11}(r) \right]^2 . \qquad (49c)$$

Of course, (49b) follows from (49a) with $\mu = 0$; (49c) is obtained from (49b) by using local isotropy and incompressibility [i.e., using (17) and (19)]. Obukhov (1949), Obukhov and Yaglom (1951), and Batchelor (1951) obtained (49c) directly from their versions of (13).

In the formulation by Hill (1993), one cannot pass to a single term as in (49c); he therefore defined the universal constant H_P such that

$$D_P(r) = H_P D_{1111}(r) , \qquad (50)$$

where r is in the inertial range. From (29) and (49a,b), we have the JGA corresponding constant

$$H_P^{JG} = \left[g(1+g) + 2 H_{\lambda\lambda}^{JG} - 6(1+g) H_{1\gamma}^{JG} \right] \Big/ 3g(1-g) = g/6(1-g) \qquad (51a)$$

$$\approx \frac{5}{3} + 3 H_{\lambda\lambda}^{JG} - 15 H_{1\gamma}^{JG} \qquad (51b)$$

$$\approx \frac{5}{3} + \frac{16}{3} - \frac{20}{3} \approx \frac{1}{3} . \qquad (51c)$$

In passing from (51b) to (51c), we use (34a,b). Of course, (51a-c) are just the formulas (49a-c) after division by $D_{1111}^{JG}(r) = 3 [D_{11}(r)]^2$, as required by the definition of H_P in (50).

The inertial range of the spectrum $\Psi_P^{JG}(k_1)$ can be obtained from (14). However, it is easier to apply the JGA to the corresponding inertial-range formula for $\Psi_P(k_1)$ given by Hill (1993) and Hill and Wilczak (1995). For simplicity, we take $C_1 = 2$ and $g = 2/3$ to obtain

$$\Psi_P^{JG}(k_1) = 3.9 H_P^{JG} \varepsilon^{4/3} k_1^{-7/3} . \qquad (52)$$

George et al. (1984) compared measurements of pressure spectra with predictions from the JGA. In the inertial range, their measured spectrum is about twice the JGA prediction.

14

The formula for mean-squared pressure gradient χ by Hill (1993) and Hill and Wilczak (1995) depends on $A_D(r)$ [defined in (9)], which has three terms arising from the three components $D_{1111}(r)$, $D_{\lambda\lambda\lambda\lambda}(r)$, and $D_{11\gamma\gamma}(r)$; they express χ in the form

$$\chi = 4 H_\chi \int_0^\infty r^{-3} D_{1111}(r) \, dr , \qquad (53)$$

where

$$H_\chi = \frac{\int_0^\infty r^{-3} A_D(r) \, dr}{\int_0^\infty r^{-3} D_{1111}(r) \, dr} . \qquad (54)$$

By applying the JGA to (53) and (54) or by using (10) and (15), we have

$$\chi^{JG} = 12 H_\chi^{JG} \int_0^\infty r^{-3} \left[D_{11}(r) \right]^2 dr , \qquad (55)$$

with

$$H_\chi^{JG} = \frac{\int_0^\infty r^{-3} \left\{ [D_{11}(r)]^2 + [D_{\lambda\lambda}(r)]^2 - 2 D_{11}(r) \, D_{\gamma\gamma}(r) \right\} dr}{\int_0^\infty r^{-3} \left[D_{11}(r) \right]^2 dr} . \qquad (56a)$$

Substituting (10) in (56a) gives

$$H_\chi^{JG} = \frac{\dfrac{1}{4} \int_0^\infty r^{-1} \left[D_{11}^{(1)}(r) \right]^2 dr}{\int_0^\infty r^{-3} \left[D_{11}(r) \right]^2 dr} . \qquad (56b)$$

15

Batchelor (1951) defined the pressure length scale λ_P from

$$\lambda_P^2 \equiv \sigma_{11}^2 / <P_{|1}^2> = 3\,\sigma_{11}^2/\chi \;,$$

from which we have the JGA version

$$\lambda_P^{JG\,2} = 3\,\sigma_{11}^2 / \chi^{JG} \;.$$

First, we consider the limit of low Reynolds numbers. In this case, we can approximate (c.f., Fig. 5.2 by Batchelor, 1956) $R_{11}(r) \approx \sigma_{11}\exp(-r^2/2\,\lambda_T^2)$ and, hence, $D_{11}(r) = 2\,\sigma_{11}[1 - \exp(-r^2/2\,\lambda_T^2)]$, where λ_T is Taylor's length scale given from $\lambda_T^2 = \sigma_{11}/d_1$. The integrals in (56b) can be performed by elementary methods in this case, yielding

$$H_\chi^{JG} = (4\ln 2)^{-1} = 0.361 \;.$$

From (55), we obtain in the limit of very low Reynolds numbers

$$\chi^{JG} = 6\,\sigma_{11}^2\,\lambda_T^{-2} = 6\,\sigma_{11}\,d_1 \;,$$

and, therefore,

$$\frac{\lambda_T^2}{\lambda_P^{JG\,2}} = 2 \;,$$

which is the same value as obtained by Uberoi (1954).

For the case of high Reynolds numbers, we closely follow the development of Hill (1993). We next examine whether or not the JGA predicts close cancellation of the three terms in H_χ^{JG}, and thereby whether or not H_χ is likely to be difficult to measure. To establish a value for H_χ^{JG} for large Reynolds numbers, we define $N_{\alpha\beta}^{JG}$ such that

$$N_{\alpha\beta}^{JG} \equiv \int_0^\infty r_{\alpha\beta}^{-3}\,\tilde{D}_{\alpha\alpha}(r_\alpha)\,\tilde{D}_{\beta\beta}(r_\beta)\,dr_{\alpha\beta} \;, \tag{57}$$

16

where $r_\alpha \equiv r/\ell_\alpha = r_{\alpha\beta}(\ell_\beta/\ell_\alpha)^{1/2}$, and similarly for r_β, and where

$$\tilde{D}_{\alpha\alpha}(r_\alpha) \equiv D_{\alpha\alpha}(r)\big/C_\alpha \varepsilon^{2/3}\ell_\alpha^g .$$

In the inertial and viscous ranges, the asymptotic formulas for $\tilde{D}_{\alpha\alpha}(r_\alpha)$ are r_α^g and r_α^2, respectively. Using (57), we find the integrals that appear in (56a) are

$$\int_0^\infty r^{-3} D_{\alpha\alpha}(r) D_{\beta\beta}(r)\, dr = C_\alpha C_\beta\, \varepsilon^{4/3} (\ell_\alpha \ell_\beta)^{g-1} N_{\alpha\beta}^{JG} . \tag{58}$$

We estimate $N_{\alpha\beta}^{JG}$ by using the ad hoc formula

$$\tilde{D}_{\alpha\alpha}(r_\alpha) = r_\alpha^2 (1 + r_\alpha^2)^{(g-2)/2} , \tag{59}$$

which interpolates between the inertial- and viscous-range asymptotes. Evaluating (57) by use of (59) gives

$$N_{\alpha\beta}^{JG} \propto \begin{cases} \dfrac{3}{2}, & \text{for } \alpha = \beta , \\[2ex] \dfrac{3}{2} - 0.01, & \text{for } \alpha \neq \beta . \end{cases} \tag{60}$$

Substituting (58) in (56a) and using (23), (33), (39), and (42), we have

$$H_\chi^{JG} = 1 + m_{\lambda\lambda}^{JG} h_{\lambda\lambda}^{JG} - 6 m_{1\gamma}^{JG} h_{1\gamma}^{JG} , \tag{61}$$

where the $m_{\alpha\beta}^{JG}$ are defined by

$$m_{\alpha\beta}^{JG} \equiv N_{\alpha\beta}^{JG}\big/N_{11}^{JG} = \begin{cases} 1 , & \text{for } \alpha = \beta , \\[2ex] 1 - 0.0066, & \text{for } \alpha \neq \beta , \end{cases}$$

and (60) is used to obtain the numerical values of $m_{\alpha\beta}^{JG}$. Hill (1993) and Hill and Wilczak (1995) showed that for the new theory the $m_{\alpha\beta}$ are expected to be very close to unity because they depend only on the relative shapes of the viscous- to inertial-range transition of

17

$D_{\alpha\alpha\beta\beta}(r)$. We adopt the value $m_{\alpha\beta}^{JG} = 1$ for all α and β because the deviation from unity may be less than errors from (59). Using the values from (43a,b) in (61), we obtain

$$H_{\chi}^{JG} \approx 1 + 2.18 - 2.95$$

$$\approx 0.23 .$$

(62)

This result is the same to two decimal places whether we take $\mu = 0$ or $\mu = 0.25$. Thus H_{χ}^{JG} is only 8% of the largest magnitude term in (62). The near cancellation of the three terms in (62) implies near cancellation of the three terms in (56a). We therefore expect that H_{χ} is difficult to measure.

For large Reynolds numbers, we can now estimate the mean-squared pressure gradient χ^{JG} in the JGA. From (55), (56a), and (58), we have

$$\chi^{JG} = 4 N_{11}^{JG} H_{\chi}^{JG} (3 C_1^2) \varepsilon^{4/3} (\ell_1)^{2g-2}$$

(63a)

$$\approx 3.0 \nu^{-1/2} \varepsilon^{3/2} .$$

(63b)

The corresponding formula for χ by Hill (1993) and Hill and Wilczak (1995) also gives (63a) if the JGA is applied to every factor in their formula. Yaglom (1949) also obtained (63b). Batchelor (1951) obtained (63b) with a somewhat different estimate for the coefficient.

Batchelor (1951) defined the pressure inner scale ℓ_p as the spacing at which the inertial-range and viscous-range asymptotic formulas for $D_p(r)$ are equal. The relationship given by Hill and Wilczak (1995) of ℓ_p to ℓ_{11} becomes, in the JGA,

$$\ell_p^{JG} = \left(\frac{3 H_p^{JG}}{4 N_{11}^{JG} H_{\chi}^{JG}} \right)^{1/(2 - 2g)} \quad \ell_{11}^{JG} = 0.74 \, \ell_1$$

(64a)

$$\approx 0.63 \, \ell_1 .$$

(64b)

From (25), (44), and (45a), we have $\ell_p^{JG} \approx \ell_{\lambda\lambda}^{JG} = \ell_{\lambda} \approx 9.6 \eta$. Therefore, in the JGA the pressure inner scale is commensurate with the inner scales of the second-order and fourth-order velocity structure functions.

The initial decrease of the pressure-gradient correlation is described by the scale λ_{α}. Specifically, Hill (1993) and Hill and Wilczak (1995) gave

18

$$A_{\alpha\alpha}(r) = \frac{\chi}{3}\left(1 - \frac{r^2}{2\lambda_\alpha^2} + \dots\right),$$

where, using definitions (36) and (46a),

$$\lambda_1^2 = \chi/36 \, d_{11} \, h_Q$$

and $\lambda_\lambda = \sqrt{3}\,\lambda_1$. In the JGA, we use (38), (46c), and (63a,b) to obtain for large Reynolds numbers

$$\lambda_1^{JG} = 0.34\,\ell_1 = 4.4\,\eta \,. \tag{65}$$

By using (64b) in the inertial-range formulas for $A_{\alpha\alpha}(r)$ as given by Hill (1993) and Hill and Wilczak (1995), we obtain

$$A_{\lambda\lambda}^{JG}(r) \approx 0.5\,\varepsilon^{4/3}\,r^{-2/3} \tag{66}$$

and $A_{11}^{JG}(r) \approx A_{\lambda\lambda}^{JG}(r)/3$. Hill and Wilczak (1995) estimated that if $\ell_P/\lambda_1 < 2.7$, then $A_{11}(r)$ has negative values somewhere between $r = \lambda_1$ and the inertial range. From (64a,b) and (65), we see that $\ell_P^{JG}/\lambda_1^{JG} < 2.2$; thus we expect that $A_{11}^{JG}(r)$ does have negative values, as obtained by Obukhov and Yaglom (1951). However, whether or not the exact function $A_{11}(r)$ has negative values remains unknown.

5.   SUMMARY AND CONCLUSIONS

A detailed examination of the application of the assumption of joint Gaussian velocities to the pressure structure function $D_P(r)$ is given. Hill (1993) and Hill and Wilczak (1995) derived $D_P(r)$ without use of this assumption. Applying the JGA to the more exact formulas by Hill (1993) and Hill and Wilczak (1995) gives the same formulas for $Q^{JG}(r)$, $D_P^{JG}(r)$, the inertial range of $D_P^{JG}(r)$, and χ^{JG} as are given by Batchelor (1951) and Obukhov and Yaglom (1951), thereby validating the results by Hill (1993) and Hill and Wilczak (1995). Simpler formulas are obtained for the pressure-gradient correlation than those given by Obukhov and Yaglom (1951).

The JGA gives specific predictions for the mean-squared pressure gradient (63b), the dissipation range of the pressure structure function (48a), and the inertial ranges of both the pressure structure function (49a-c) and the pressure-gradient correlation (66).

19

The JGA predicts values of constants and parameters that are defined by Hill (1993). Specifically, the ratio of components of the fourth-order velocity structure function $D_{\alpha\alpha\beta\beta}(r)$ in the inertial range are $H_{\lambda\lambda}^{JG} \approx 16/9$ and $H_{1\lambda}^{JG} \approx 4/9$ and in the dissipation range are $\Lambda_{\lambda\lambda}^{JG} = 4$ and $\Lambda_{1\lambda}^{JG} = 2/3$; the constant $h_Q^{JG} = 1/3$. These values give $H_P^{JG} \approx 1/3$ and, for high Reynolds numbers, $H_\chi^{JG} \approx 0.23$, which are respectively 5% and 8% of the largest term in (51c) and (62). This close cancellation of terms suggests that the constants H_P and H_χ (without the JGA) will be difficult to measure. For very low Reynolds numbers, $H_\chi^{JG} = 0.361$.

The inner scales of the second-order velocity structure function and the JGA predictions for inner scales of fourth-order velocity structure functions and the pressure structure function are found to have similar values, and the correlation scales λ_α^{JG} of the pressure-gradient correlation are, of course, smaller; specifically, $\ell_P \approx \ell_\lambda = 0.74\,\ell_1 = 0.74\,\ell_{11}^{JG} = \ell_{\lambda\lambda}^{JG} = \sqrt{0.74}\,\ell_{1\lambda}^{JG} = 2.2\,\lambda_\lambda = 3.8\,\lambda_1$. The accuracy of specific predictions of the JGA can be tested using measurements of the velocity vector; pressure measurements are not needed for locally isotropic turbulence in incompressible fluid.

When the intermittency parameter μ is included in inertial-range exponents, the JGA gives $q^{JG} = (4/3) + (2\mu/9)$ for the power-law exponent of the pressure structure function as well as for components of the fourth-order velocity structure function. On the other hand, the more exact theory gives $q = (4/3) - (2\mu/9)$. The difference, $q^{JG} - q = 4\mu/9$, emphasizes that the JGA is incompatible with intermittency. Use of the JGA in combination with the intermittency-theory (and, presumably, most accurate) formula for $D_{ij}(\vec{r})$ causes less accurate pressure statistics than does the combination of the JGA and the classic theory of $D_{ij}(\vec{r})$ given by Kolmogorov (1941). The simpler case of $\mu = 0$ is therefore preferred in formulas from the JGA.

Using the JGA, we estimate that the longitudinal component of the pressure-gradient correlation, $A_{11}^{JG}(r)$, has negative values somewhere between $r = \lambda_1$ and the inertial range. On the basis of the JGA and a specific model for $D_{11}(r)$, Obukhov and Yaglom (1951) also obtained such negative values for spacings $r = 1.3\,\ell_1$ to $r = 2.5\,\ell_1$. However, the exact $A_{11}(r)$ might not have negative values.

6.   ACKNOWLEDGMENT

7.   REFERENCES

Batchelor, G. K., 1951. Pressure fluctuations in isotropic turbulence. *Proc., Camb. Philos. Soc.* **47**:359-374.

Batchelor, G. K., 1956. *The Theory of Homogeneous Turbulence.* Cambridge University Press, Cambridge, England, 197 pp.

George, W. K., P. D. Beuther, and R. E. A. Arndt, 1984. Pressure spectra in turbulent free shear flows. *J. Fluid Mech.* **148**:155-191.

Hill, R. J., 1993. Relationships between several fourth-order velocity statistics and the pressure structure function for isotropic, incompressible turbulence. NOAA Tech. Rep. ERL 449-ETL 65, NOAA/ERL Environmental Technology Laboratory, Boulder, CO (order no. PB94-187309, available from the National Technical Information Service, 5285 Port Royal Rd., Springfield, VA 22161).

Hill, R. J., and J. M. Wilczak, 1995. Pressure structure functions and spectra for locally isotropic turbulence. *J. Fluid Mech.*, accepted.

Kolmogorov, A. N., 1941. The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers. *Dokl. Akad. Nauk. SSSR* **30**:299-303.

Obukhov, A. M., 1949. Pressure fluctuations in turbulent flow. *Dokl. Akad. Nauk. SSSR* **66**:17-20.

Obukhov, A. M., and A. M. Yaglom, 1951. The microstructure of turbulent flow. *Prikl. Mat. Mekh.* **15**:3-26 (translation in NACA TM 1350, June 1953, National Advisory Committee for Aeronautics).

Praskovsky, A., and S. Oncley, 1994. Measurements of the Kolmogorov constant and intermittency exponent at very high Reynolds numbers. *Phys. Fluids.* **6**:2886-2888.

Sreenivasan, K. R., and P. Kailasnath, 1993. An update on the intermittency exponent in turbulence. *Phys. Fluids* **A5**:512-514.

Uberoi, M. S., 1954. Correlations involving pressure fluctuations in homogeneous turbulence. NACA TM 3116, National Advisory Committee for Aeronautics, 61 pp.

Yaglom, A. M., 1949. On the acceleration field in a turbulent flow. *Dokl. Akad. Nauk. SSSR* **67**:795-798.

Yaglom, A. M., 1981. Laws of small-scale turbulence in atmosphere and ocean (in commemoration of the 40th anniversary of the theory of locally isotropic turbulence). *Izv. Acad. Sci. USSR Atmos. Oceanic Phys.* **17**:919-935.